Combinatorics

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Summary

Permutations

Definition

The arrangement of different objects into a linear order using each object exactly once is called a *permutation* of these objects. The number $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ of all permutations of n objects is called n *factorial*, and is denoted by $n!$.

Permutations

Theorem

We operate by the convention that $0! \equiv 1$. If we assume n people arrive at a dentist's office and the dentist treats them one by one, how many different orders that each patient will be served are possible?

Permutations

Theorem

The number of orders in which the patients can be treated is $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$.

Example

If 8 people enter a dentist's office the dentist treats them one by one, there are $8! = 40320$ different orders that each patient will be served.

Partitions

Definition

Let $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ be integers so that $a_1 + a_2 + \cdots + a_k = \text{n}$. Then sequence $(a_1, a_2, \cdots, a_k)$ is called a *partition* of the integer n. The number of all partitions is denoted by $p(n)$. The number of partitions of n into exactly k parts is denoted by $p_k(n)$.

Partitions

$$n = integer$$

$$k = parts$$

Example

If n = 5 and k = 2 a possible partition would be (3+2) or (4+1)

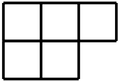
Ferrers Shape

Definition

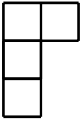
For partition of integers there is an array of cells with with n_i-cells in the ith row. So, when reading the diagram horizontally, a_1 will correspond with the 1st row, and so on.

Let P be the partition of 5; Ferrers shape for P is:

$$P = 3 + 2$$



$$P = 2 + 1 + 1$$

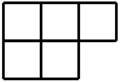


Ferrers Shape

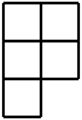
Conjugation

The *conjugation* of a partition can be found by reading Ferrers shape vertically opposed to horizontally.

$$P = 3 + 2$$



The conjugate of P of 5 is:



$$P' = 2 + 2 + 1$$

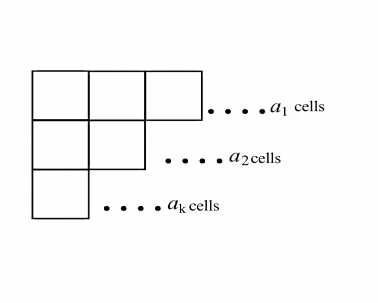
Ferrers Shape

Theorem

The number of partitions into at most k parts is equal to that of partition of n into parts not larger than k.

Proof

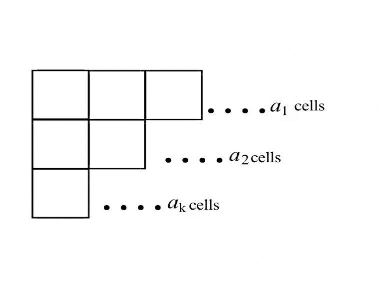
First, consider the partition of n into k-parts:



Ferrers Shape

Proof

Then, consider the conjugation of the partition of n into k-parts: We know every row has to have at least one cell, so when you begin to count vertically, the number of parts we initially split n into will become the first value, and therefore the greatest value, of our conjugate.



Ferrers Shape

The partition of 8 into 3-parts is:

$$(6 + 1 + 1)$$

$$(5 + 2 + 1)$$

$$(4 + 3 + 1)$$

$$(4 + 2 + 2)$$

$$(3 + 3 + 2)$$

The partition of 8 into parts the largest size of which is 3 is:

$$(3 + 1 + 1 + 1 + 1 + 1)$$

$$(3 + 2 + 1 + 1 + 1)$$

$$(3 + 2 + 2 + 1)$$

$$(3 + 3 + 1 + 1)$$

$$(3 + 3 + 2)$$

Binomial Theorem

Definition

A binomial coefficient is defined as, for nonnegative integers n and k with $n \geq k$, the expression

$$\frac{n!}{k!(n-k)!} = \binom{n}{k},$$

where $n! = 1 \cdot 2 \cdot 3 \ldots (n-1) \cdot n$.

Binomial Theorem

For all nonnegative integers n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Pascal's Triangle

n=0								1						
n=1							1		1					
n=2						1		2		1				
n=3					1		3		3		1			
n=4				1		4		6		4		1		
n=5			1		5		10		10		5		1	
n=6		1		6		15		20		15		6		1

For all nonegative integers n and k,

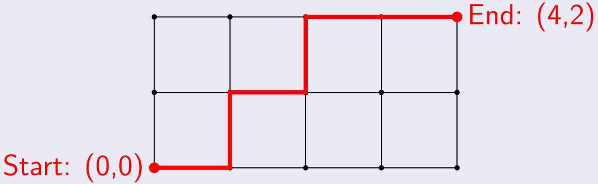
$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Lattice Paths Intro

A northeast lattice path is a path along the Cartesian plane that uses the steps (1,0) or (0,1).

Example



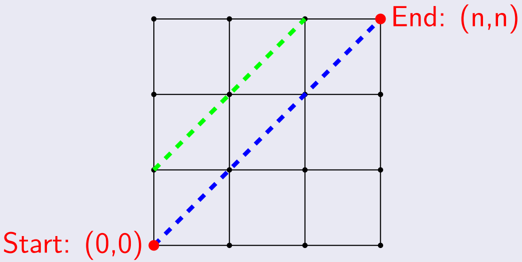
How many northeast lattice paths are there from (0,0) to (4,2)?

$$\binom{6}{2} = \boxed{15}.$$

Harder Lattice Path Example

Example

How many northeastern lattice paths from $(0,0)$ to (n,n) never go above the diagonal $x = y$?



Definition

A "bad" lattice path will be one that goes through the diagonal $x = y$; in other words, the path touches the line $x = y + 1$.

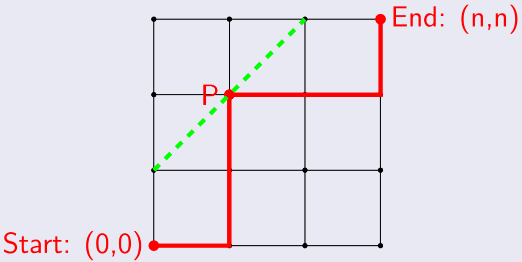
Example Continued

Question

How many northeastern lattice paths from $(0, 0)$ to (n, n) never go above the diagonal $x = y$?

Solution

We can reflect the segment of our path from the origin $(0, 0)$ to point P along the line $y = x + 1$.



Lattice Paths Solution

Question

How many northeastern lattice paths from $(0, 0)$ to (n, n) never go above the diagonal $x = y$?

Number of "bad" paths:

$$\binom{n-1+n+1}{n-1} = \binom{2n}{n-1}.$$

Total number of paths:

$$\binom{2n}{n} - \binom{2n}{n-1}.$$

References

M. Bona *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory*, 2017.

Acknowledgements

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